ON HERMITE HADAMARD INEQUALITIES FOR PRODUCT OF TWO log-φ-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce the notion of log-φ-convex functions and present some properties and representation of such functions. We obtain some results of the Hermite Hadamard inequalities for product log-φ-convex functions.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[4],[8, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[10]) and the references cited therein.

The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

A function $f : I \to [0, \infty)$ is said to be log-convex or multiplicatively convex if $\log t$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.2}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a *log*-convex function is convex, but the converse may not necessarily be true [7]. This follows directly from (1.2) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results related to this classical results, (see[4],[5],[9],[10]) and the references therein. Dragomir and Mond [6] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

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$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left[f\left(x\right)\right]dx\right] \tag{1.3}$$
$$\leq \frac{1}{b-a}\int_a^b G\left(f\left(x\right),f\left(a+b-x\right)\right)dx$$
$$\leq \frac{1}{b-a}\int_a^b f\left(x\right)dx$$
$$\leq L\left(f\left(a\right),f\left(b\right)\right)$$
$$\leq \frac{f\left(a\right)+f\left(b\right)}{2},$$

where $G\left(p,q\right)=\sqrt{pq}$ is the geometric mean and $L\left(p,q\right)=\frac{p-q}{\ln p-\ln q}$ $\left(p\neq q\right)$ is the logarithmic mean of the positive real numbers p,q (for $p=q,$ we put $L\left(p,q\right)=p$).

Let us consider a function $\varphi:[a,b]\rightarrow[a,b]$ where $[a,b]\subset\mathbb{R}$. Youness have defined the φ-convex functions in [11]:

Definition 1. *A function $f:[a,b]\rightarrow\mathbb{R}$ is said to be φ- convex on $[a,b]$ if for every two points $x\in[a,b], y\in[a,b]$ and $t\in[0,1]$ the following inequality holds:*

$$f(t\varphi(x)+(1-t)\varphi(y))\leq tf(\varphi(x))+(1-t)f(\varphi(y)).$$

In [2], Cristescu proved the followig results for the φ-convex functions

Lemma 1. *For $f:[a,b]\rightarrow\mathbb{R}$, the following statements are equivalent:*
(i) f is φ-convex functions on $[a,b]$,
(ii) for every $x,y\in[a,b]$, the mapping $g:[0,1]\rightarrow\mathbb{R},\ g(t)=f(t\varphi(x)+(1-t)\varphi(y))$ is classically convex on $[0,1]$.

Obviously, if function φ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the φ-convex functions can be found, for instance, in [1], [2],[11].

In this paper, we introduce the notion of log-φ-convex functions and we obtain a representation of log-φ-convex. Finally, a version of Hermite–Hadamard-type inequalities for log-φ-convex functions is presented.

2. Main Results

Let us consider a $\varphi:[a,b]\rightarrow[a,b]$ where $[a,b]\subset\mathbb{R}$ and I stands for a convex subset of $\mathbb{R}$. We say that a function $f:I\rightarrow\mathbb{R}^{+}$ is a log-φ-convex if

$$f(t\varphi(x)+(1-t)\varphi(y))\leq\left[f(\varphi(x))\right]^{t}\left[f(\varphi(y))\right]^{1-t} \tag{2.1}$$

for all $x,y\in I$ and $t\in[0,1]$. We say that f is a log-φ-midconvex if (2.1) is assumed only for $t=\frac{1}{2}$, that is

$$f\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\leq\sqrt{f(\varphi(x))(f(\varphi(y))},\ \text{for } x,y\in I$$

Obviously, if function φ is the identity, then the classical logarithmic convexity is obtained from (2.1).

From the above definitions, we have

$$\begin{aligned} f(t\varphi(x)+(1-t)\varphi(y)) &\leq [f(\varphi(x))]^{t}\,[f(\varphi(y))]^{1-t} \\ &\leq tf(\varphi(x))+(1-t)f(\varphi(y)) \\ &\leq \max\{f(\varphi(x)),f(\varphi(y))\}. \end{aligned}$$

Lemma 2. *For $f:[a,b]\to\mathbb{R}^{+}$, the following statements are equivalent:*
(i) f is log-*φ-convex functions on $[a,b]$,*
(ii) for every $x,y\in[a,b]$, the mapping

$$g:[0,1]\to\mathbb{R}^{+},\ g(t)=f(t\varphi(x)+(1-t)\varphi(y))$$

is classically log-*convex on $[0,1]$.*

Proof. Let us consider two points $x,y\in[a,b]$, $\lambda\in[0,1]$ and $t_1,t_2\in[0,1]$. Then, we obtain

$$\begin{aligned} &g(\lambda t_1+(1-\lambda)t_2) \\ =\ & f([\lambda t_1+(1-\lambda)t_2]\,\varphi(x)+[1-\lambda t_1-(1-\lambda)t_2]\,\varphi(y)) \\ =\ & f(\lambda\,[t_1\varphi(x)+(1-t_1)\varphi(y)]+(1-\lambda)\,[t_2\varphi(x)+(1-t_2)\varphi(y)]) \\ \leq\ & [f(t_1\varphi(x)+(1-t_1)\varphi(y))]^{\lambda}\,[f\,(t_2\varphi(x)+(1-t_2)\varphi(y))]^{1-\lambda} \\ =\ & [g(t_1)]^{\lambda}\,[g(t_2)]^{1-\lambda} \end{aligned}$$

which gives that g is log-convex function.

Conversely, if g is log-convex function for $x,y\in[a,b]$, $\lambda\in[0,1]$ and $t_1=1,t_2=0$, then we get

$$\begin{aligned} f(\lambda\varphi(x)+(1-\lambda)\varphi(y)) &= g(\lambda 1+(1-\lambda)0)) \\ &\leq [g(1)]^{\lambda}\,[g(0)]^{1-\lambda} \\ &= [f(\varphi(x))]^{\lambda}\,[f(\varphi(y))]^{1-\lambda} \end{aligned}$$

which shows that f is log-φ-convex. This completes to proof. □

We give now a new Hermite–Hadamard-type inequalities for log-φ-convex functions:

Theorem 1. *If $f:[a,b]\to\mathbb{R}^{+}$ is* log-φ*-convex for the continuous function $\varphi:[a,b]\to[a,b]$, then*

$$
\begin{aligned}
(2.2)\quad f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &\leq \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} G\left(f(x),f(\varphi(a)+\varphi(b)-x)\right)dx \\
&\leq \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx \\
&\leq \frac{f(\varphi(b))-f(\varphi(a))}{\log f(\varphi(b))-\log f(\varphi(a))}=L\left(f(\varphi(b)),f(\varphi(a))\right) \\
&\leq \frac{f(\varphi(a))+f(\varphi(b))}{2}.
\end{aligned}
$$

Proof. Since f be log–φ-convex functions, we have that for all $t\in[0,1]$

$$
\begin{aligned}
f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &= f\left(\frac{t\varphi(a)+(1-t)\varphi(b)}{2}+\frac{(1-t)\varphi(a)+t\varphi(b)}{2}\right) \\
&\leq \sqrt{\left[f(t\varphi(a)+(1-t)\varphi(b))\right]\left[f((1-t)\varphi(a)+t\varphi(b))\right]}
\end{aligned}
$$

Integrating the above inequality with respect to t over $[0,1]$ and we also use the substitution $x=(1-t)\varphi(a)+t\varphi(b)$, we obtain

$$
\begin{aligned}
& f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\
\leq\ & \int\limits_{0}^{1} \sqrt{\left[f(t\varphi(a)+(1-t)\varphi(b))\right]\left[f((1-t)\varphi(a)+t\varphi(b))\right]}dt \\
=\ & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} \sqrt{f(x)f(\varphi(a)+\varphi(b)-x)}dx \\
\leq\ & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} A\left(f(x),f(\varphi(a)+\varphi(b)-x)\right)dx
\end{aligned}
$$

and so for

$$
\int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx=\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(a)+\varphi(b)-x)dx
$$

$$\begin{aligned}
(2.3)\qquad & f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\
\leq\ & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} G\left(f(x), f(\varphi(a)+\varphi(b)-x)\right)dx \\
\leq\ & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx.
\end{aligned}$$

From the log-φ-convexity of f, we have

$$\begin{aligned}
(2.4)\qquad & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx \\
=\ & \int\limits_0^1 f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt \\
\leq\ & \int\limits_0^1 \left[f(\varphi(a))\right]^t\left[f(\varphi(b))\right]^{1-t}dt \\
=\ & f(\varphi(b))\int\limits_0^1 \left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t dt \\
=\ & f(\varphi(b))\frac{1}{\log f(\varphi(a))-\log f(\varphi(b))}\left[\frac{f(\varphi(a))}{f(\varphi(b))}-1\right] \\
=\ & \frac{f(\varphi(b))-f(\varphi(a))}{\log f(\varphi(b))-\log f(\varphi(a))} = L\left(f(\varphi(b)), f(\varphi(a))\right) \\
\leq\ & \frac{f(\varphi(a))+f(\varphi(b))}{2}.
\end{aligned}$$

Thus, from (2.3) and (2.4) we obtain required result (2.2). This completes to proof. $\square$

Theorem 2. *If $f, g : [a,b] \to \mathbb{R}^+$ is* log-φ- *convex for the continuous function $\varphi : [a,b] \to [a,b]$, then*

$$\begin{aligned}
(2.5)\qquad & \frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(x)\,g(x)\,dx \leq L\left(f(\varphi(b))g(\varphi(b)), f(\varphi(a))g(\varphi(a))\right) \\
\leq\ & \frac{1}{4}\left\{\left(\left[f(\varphi(b))\right]+\left[f(\varphi(a))\right]\right)L(\left[f(\varphi(b))\right],\left[f(\varphi(a))\right])\right\} \\
& +\frac{1}{4}\left\{\left(\left[g(\varphi(b))\right]+\left[g(\varphi(a))\right]\right)L(\left[g(\varphi(b))\right],\left[g(\varphi(a))\right])\right\}.
\end{aligned}$$

Proof. Since f and g be log–φ-convex functions, we have that for all $t\in[0,1]$

$$f(t\varphi(a)+(1-t)\varphi(b))\leq [f(\varphi(a))]^{t}[f(\varphi(b))]^{1-t}$$

and

$$g(t\varphi(a)+(1-t)\varphi(b))\leq [g(\varphi(a))]^{t}[g(\varphi(b))]^{1-t}.$$

Thus, it follows that

$$\begin{aligned}
& \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\,g(x)\,dx \\
& \int_0^1 f(t\varphi(a)+(1-t)\varphi(b))g(t\varphi(a)+(1-t)\varphi(b))dt \\
\leq\ & \int_0^1 [f(\varphi(a))]^{t}[f(\varphi(b))]^{1-t}[g(\varphi(a))]^{t}[g(\varphi(b))]^{1-t}\,dt \\
=\ & f(\varphi(b))g(\varphi(b))\int_0^1\left[\frac{f(\varphi(a))g(\varphi(a))}{f(\varphi(b))g(\varphi(b))}\right]^{t}dt \\
=\ & \frac{f(\varphi(b))g(\varphi(b))}{\log f(\varphi(a))g(\varphi(a))-\log f(\varphi(b))g(\varphi(b))}\left[\frac{f(\varphi(a))g(\varphi(a))}{f(\varphi(b))g(\varphi(b))}-1\right] \\
=\ & \frac{f(\varphi(b))g(\varphi(b))-f(\varphi(a))g(\varphi(a))}{\log f(\varphi(b))g(\varphi(b))-\log f(\varphi(a))g(\varphi(a))} \\
=\ & L\left(f(\varphi(b))g(\varphi(b)),f(\varphi(a))g(\varphi(a))\right) \\
\leq\ & \frac{1}{2}\int_0^1\left([f(t\varphi(a)+(1-t)\varphi(b))]^{2}+[g(t\varphi(a)+(1-t)\varphi(b))]^{2}\right)dt \\
\leq\ & \frac{1}{2}\int_0^1\left([f(\varphi(a))]^{2t}[f(\varphi(b))]^{2-2t}+[g(\varphi(a))]^{2t}[g(\varphi(b))]^{2-2t}\right)dt \\
=\ & \frac{1}{4}\left\{[f(\varphi(b))]^{2}\int_0^2\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^{u}du+[g(\varphi(b))]^{2}\int_0^2\left[\frac{g(\varphi(a))}{g(\varphi(b))}\right]^{u}du\right\} \\
=\ & \frac{1}{4}\left\{\frac{[f(\varphi(b))]^{2}-[f(\varphi(a))]^{2}}{\log f(\varphi(b))-\log f(\varphi(a))}+\frac{[g(\varphi(b))]^{2}-[g(\varphi(a))]^{2}}{\log g(\varphi(b))-\log g(\varphi(a))}\right\} \\
=\ & \frac{1}{4}\left\{([f(\varphi(b))]+[f(\varphi(a))])\,L([f(\varphi(b))],[f(\varphi(a))])\right\} \\
& +\frac{1}{4}\left\{([g(\varphi(b))]+[g(\varphi(a))])\,L([g(\varphi(b))],[g(\varphi(a))])\right\}
\end{aligned}$$

which is the required (2.5). This proves the theorem. $\square$

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